10.1 Introduction to Differential Equations

#1 $\frac{dy}{dx} = \frac{2x}{y}$

① Separate the variables

$y \cdot \frac{dy}{dx} = 2x$

$y\,dy = 2x\,dx$

② Integrate both sides

$\int y\,dy = \int 2x\,dx$

(constant needed on only one side)

$\frac{1}{2}y^2 = \frac{\cancel{2}x^2}{\cancel{2}} + c$

$y^2 = 2x^2 + C$

$y = \pm\sqrt{2x^2 + c}$

#2 $\frac{dy}{dt} = 0.2y$ $y(0) = 10$ initial condition

$\frac{1}{y}\frac{dy}{dt} = 0.2$

$\frac{1}{y}\,dy = 0.2\,dt$

$\int \frac{1}{y}\,dy = \int 0.2\,dt$

$\ln|y| = 0.2t + c$

solve for y: change to exponential form

$\ln|m| = n$

$e^n = m$

$e^{0.2t + c} = y$

$e^{0.2t} \cdot e^{c} = y$

$x^m \cdot x^n = x^{m+n}$

e^c is a constant

$$Ce^{0.2t} = y$$

Use the initial condition to solve for C.

$y(0) = 10$

$$Ce^{0.2(0)} = 10$$

$$Ce^{0} = 10$$

$$C = 10$$

$$y = 10e^{0.2t}$$

#3 The rate of change of y is proportional to y. When $t=0$ $y=2$ and when $t=2$ $y=4$. Find y when $t=3$

$$\frac{dy}{dt} = Ky$$

$$\frac{1}{y}\,dy = K\,dt$$

$$\int \frac{1}{y}\,dy = \int K\,dt$$

$$\ln|y| = Kt + c$$

$$e^{Kt+c} = y$$

$$e^{Kt} \cdot e^{c} = y$$

$$Ce^{Kt} = y$$

$t=0$ $y=2$

$$Ce^{K(0)} = 2$$

$$C = 2$$

$$y = 2e^{Kt}$$

$t=2$ $y=4$

$$4 = 2e^{K(2)}$$

$$2 = e^{2K}$$

$$\ln 2 = \ln e^{2K}$$

$$\ln 2 = 2K \ln e$$

$$\frac{\ln 2}{2} = K$$

$$y = 2e^{\frac{\ln 2}{2}t}$$ find y when $t = 3$

$$y = 2e^{\frac{\ln 2}{2}(3)}$$

$$y = 5.657$$

#4 $\frac{dy}{dt} = t^{-2}(y-1)$

$$\frac{1}{y-1}\,dy = t^{-2}\,dt$$

$$\int \frac{1}{y-1}\,dy = \int t^{-2}\,dt$$

$u = y-1$

$\frac{du}{dy} = 1$

$du = dy$

$$\int \frac{1}{u}\,du = \int t^{-2}\,dt$$

$$\ln|u| = \frac{t^{-1}}{-1} + c$$

$$\ln|y-1| = -\frac{1}{t} + c$$

$$e^{-\frac{1}{t}+c} = y-1$$

$$e^{-\frac{1}{t}} \cdot e^{c} + 1 = y$$

$$y = Ce^{-1/t} + 1$$

#5 $\frac{dy}{dt} = t(4-2y)$ $y(0) = 4$

$$\frac{1}{4-2y}\,dy = t\,dt$$

$$\int \frac{1}{4-2y}\,dy = \int t\,dt$$

$u = 4-2y$

$\frac{du}{dy} = -2$

$\frac{du}{-2} = dy$

$$\int \frac{1}{u} \cdot \frac{du}{-2} = \int t\,dt$$

$$-\frac{1}{2} \ln|u| = \frac{1}{2}t^2 + c$$

$$\ln|4-2y| = -t^2 + C$$

$$e^{-t^2+c} = 4-2y$$

$$e^{-t^2} \cdot e^c = 4-2y$$

$$C e^{-t^2} = 4-2y$$

$$C e^{-t^2} - 4 = -2y$$

$$C e^{-t^2} + 2 = y$$

Find C

$$y(0) = 4$$

$$4 = C e^{-(0)^2} + 2$$

$$4 = C(1) + 2$$

$$C = 2$$

$$y = 2e^{-t^2} + 2$$

check: $y = 2e^{-t^2} + 2$ take the derivative

$$\frac{dy}{dt} = 2e^{-t^2} \cdot (-2t) + 0$$

$$y = 2e^{-t^2} + 2$$

$$y - 2 = 2e^{-t^2}$$

$$\frac{dy}{dt} = (y-2)(-2t)$$

$$\frac{dy}{dt} = t(-2y+4)$$

#6 $\frac{dy}{dx} = (x-1)(y-2)$ $\qquad y(0) = 3$

$$\frac{1}{y-2}\,dy = (x-1)\,dx$$

$$\int \frac{1}{y-2}\,dy = \int (x-1)\,dx$$

$$\ln|y-2| = \frac{1}{2}x^2 - x + c$$

$$e^{\frac{1}{2}x^2 - x + c} = y - 2$$

$$C e^{\frac{1}{2}x^2 - x} + 2 = y \qquad y(0) = 3$$

$$C e^{\frac{1}{2}(0)^2 - 0} + 2 = 3$$

$$C(1) + 2 = 3$$

$$C = 1$$

$$y = e^{\frac{1}{2}x^2 - x} + 2$$