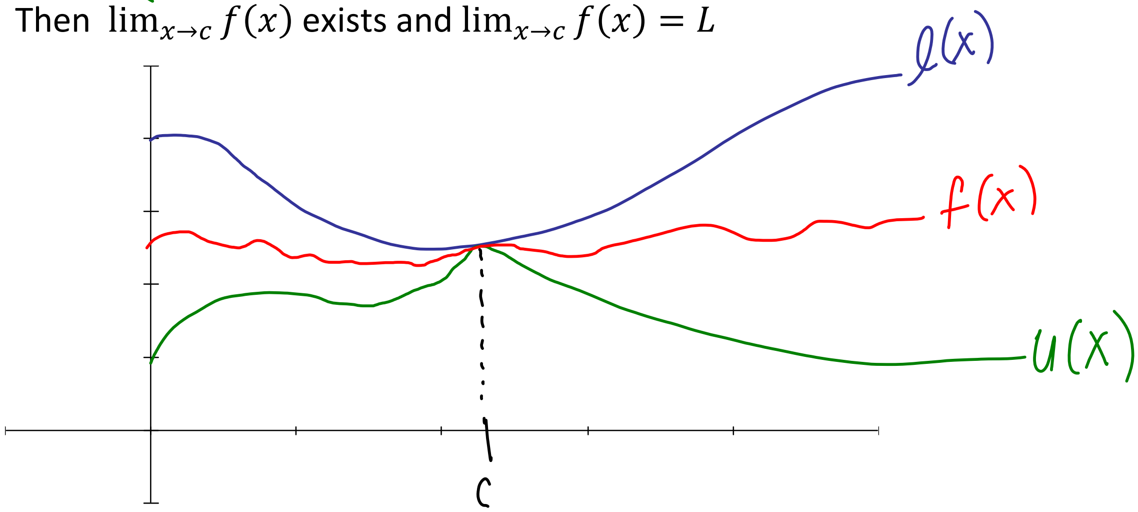
2.6 Trigonometric Limits

Squeeze Theorem:

If Assume that for $x \neq c$ (in some open interval containing c),

$l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$

Then $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = L$



1. Show that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$ using the squeeze theorem.

$-1 \leq \sin\frac{1}{x} \leq 1$

$\left|\sin\frac{1}{x}\right| \leq 1$

$\left|x\sin\frac{1}{x}\right| \leq |x|$

$-|x| \leq x\sin\frac{1}{x} \leq |x|$

$\lim_{x\to 0} -|x| = 0$

$\lim_{x\to 0} |x| = 0$

$\therefore \lim_{x\to 0} x\sin\frac{1}{x} = 0$

Special Trigonometric Limits

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

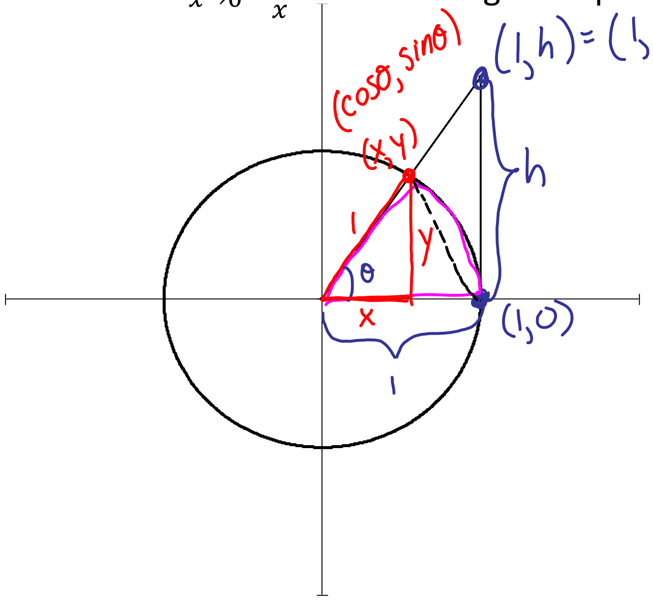
AP Calculus

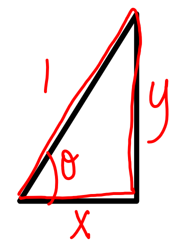
$\tan\theta = \frac{h}{1}$

$\tan\theta = h$

Proof of $\lim_{x\to 0}\frac{\sin x}{x} = 1$ using the squeeze theorem

unit circle





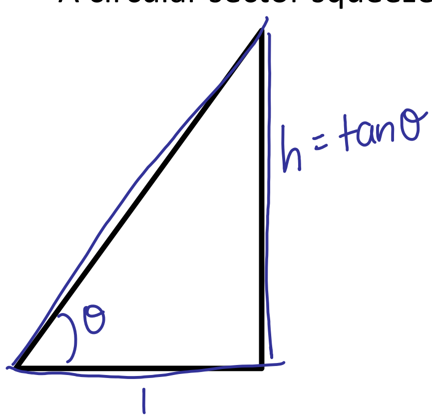
$\sin\theta = \frac{y}{1}$

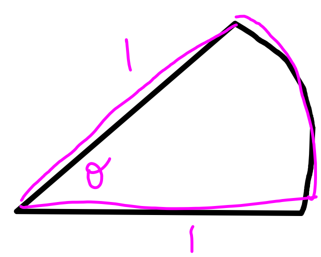
$y = \sin\theta$

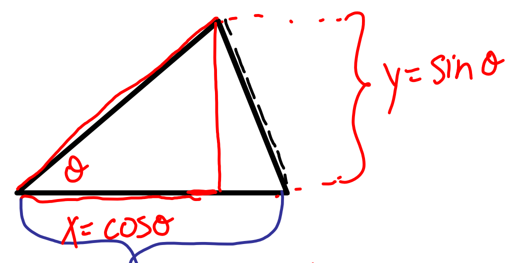
$\cos\theta = \frac{x}{1}$

$x = \cos\theta$

A circular sector squeezed between two triangles:







Area

$A = \frac{bh}{2} = \frac{1(\tan\theta)}{2}$

$A = \frac{\tan\theta}{2}$

Area

$A = \pi r^2 \cdot \frac{\theta}{2\pi}$

$A = \frac{\theta}{2}$

Area

$A = \frac{b \cdot h}{2}$

$A = \frac{1 \cdot \sin\theta}{2} = \frac{\sin\theta}{2}$

$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$

$\frac{\sin\theta}{\cos\theta} \geq \theta \geq \sin\theta$

$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$

$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$

$\lim_{\theta\to 0} \cos\theta = 1$ $\quad$ $\lim_{\theta\to 0} 1 = 1$

$\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

By the squeeze theorem.

2. Evaluate the limit $\lim_{x\to 0} \frac{\tan x}{x}$

$$\lim_{x\to 0} \frac{\sin x}{\cos x} \cdot \frac{1}{x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$\left[\lim_{x\to 0} \frac{\sin x}{x}\right] \cdot \left[\lim_{x\to 0} \frac{1}{\cos x}\right]$$

$$1 \cdot 1$$

$$= 1$$

3. Evaluate the limit $\lim_{x\to 0} \frac{\sin 4x}{x}$

$$\lim_{x\to 0} \frac{4\sin 4x}{4x}$$

$$= 4\left[\lim_{x\to 0} \frac{\sin 4x}{4x}\right]$$

$$= 4\left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right]$$

$$= 4[1]$$

$$= 4$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

$\theta = 4x$

as $x \to 0$
$4x \to 0$
$\theta \to 0$