AP Calculus

3.9 Related Rates

Given a rate of change of one quantity we are asked to find the rate of change of a related quantity.

Example #1

If $xy^3 = 24$ and $\frac{dy}{dt} = 4$ find $\frac{dx}{dt}$ when y=3.

$$x \cdot 3y^2 \frac{dy}{dt} + y^3 \cdot 1 \frac{dx}{dt} = 0$$

if $y = 3$ find x

$$xy^3 = 24$$

$$x(3)^3 = 24$$

$$x = \frac{24}{27} = \frac{8}{9}$$

$$\frac{8}{9} \cdot 3(3)^2 \cdot 4 + (3)^3 \cdot \frac{dx}{dt} = 0$$

$$27 \frac{dx}{dt} = -96$$

$$\frac{dx}{dt} = \frac{-96}{27} = \frac{-32}{9}$$

1. Identify information, Assign variables, Write an equation that relates the quantities
2. Use implicit differentiation with respect to time
3. Sub in any given values and solve for the rate specified.

Example #2

A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cm^3/min. At what rate is the radius decreasing when the radius is 5 cm.

$V = \frac{4}{3}\pi r^3$

V = Volume sphere
r = radius

$\frac{dV}{dt} = -1\ cm^3/min$

find $\frac{dr}{dt}$ when $r = 5cm$

$\frac{dV}{dt} = \frac{4\pi}{3}(3r^2)\frac{dr}{dt}$

$-1 = \frac{4\pi}{\cancel{3}}(\cancel{3})(5)^2\frac{dr}{dt}$

$\frac{-1}{4\pi(25)} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-1}{100\pi}\ cm/min$

$\frac{dr}{dt} = -.00318\ cm/min$

Example #3

A water tank is built in the shape of a circular cone with height 5m and diameter 6m at the top. Water is being pumped into the tank at a rate of 1.6m^3/min. Find the rate at which the water level is rising when the water is 2m deep.

height
$h = 5$

diameter
$d = 6$
$r = 3$

$\frac{dV}{dt} = 1.6\ m^3/min$

find $\frac{dh}{dt}$ when $h = 2$

$V = \frac{1}{3}\pi r^2 h$

$\frac{h}{r} = \frac{5}{3}$

$3h = 5r$

$\frac{3}{5}h = r$

$V = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h$

$V = \frac{1}{\cancel{3}}\pi\left(\frac{\overset{3}{\cancel{9}}}{25}h^2\right)h$

$V = \frac{3\pi}{25}h^3$

$\frac{dV}{dt} = \frac{3\pi}{25}\cdot 3h^2 \cdot \frac{dh}{dt}$

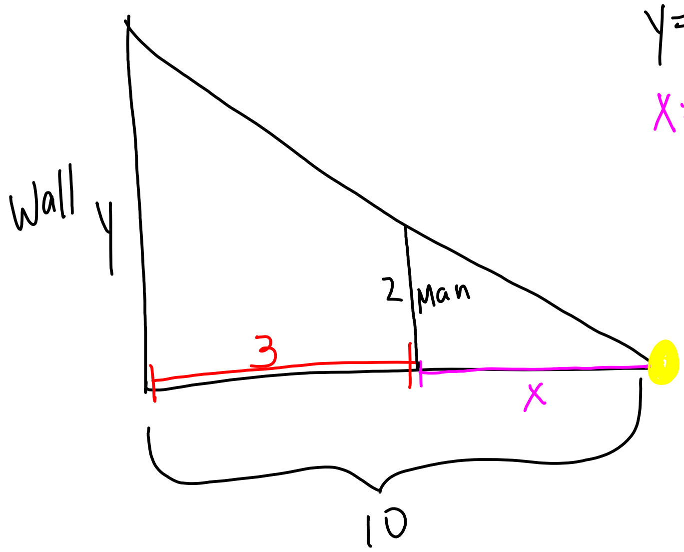
$1.6 = \frac{3\pi}{25}\cdot 3(2)^2\frac{dh}{dt}$

$\frac{1.6(25)}{3\pi(3)(4)} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{10}{9\pi}\ m/min$

Example #4

A spotlight on the ground shines on a wall 10m away. A ~~main~~ man 2m tall walks from the spotlight toward the wall at a speed of 1.2 m/s. How fast is his shadow on the wall decreasing when he is 3m from the wall?



y = size of shadow

x = distance walked

$\frac{dx}{dt} = 1.2 m/s$ speed walking

find $\frac{dy}{dt}$ when $x = 7$

3 from the wall means he walked 7m

$\frac{Big}{small}$ $\quad \frac{y}{2} = \frac{10}{x}$

$$xy = 20$$

$$y = \frac{20}{x} = 20x^{-1}$$

$$\frac{dy}{dt} = 20(-1)x^{-2}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-20}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-20}{7^2} \cdot (1.2)$$

$$dy/dt = \frac{-24}{49} m/s$$

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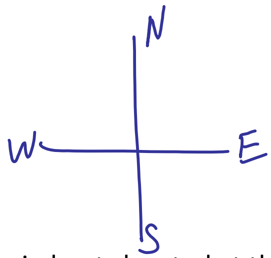
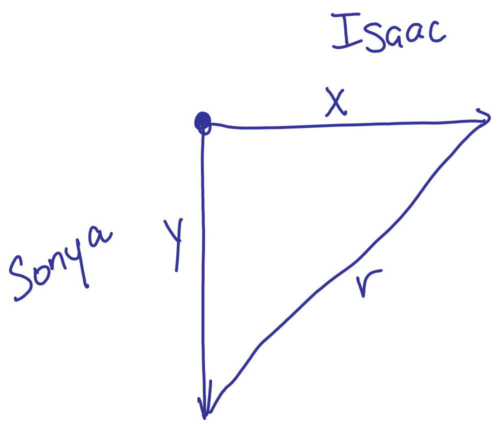


Example #5

Sonya and Isaac are in boats located at the center of a lake. At time $t=0$, Sonya begins travelling south at a speed of 32mph. At the same time Isaac takes off, heading east at a speed of 27 mph. At what rate are they separating after 12 mins.



x = distance Isaac travelled

y = distance Sonya travelled

r = distance between Isaac and Sonya

$\frac{dy}{dt} = 32 \text{ mi/h}$ $\qquad \frac{dx}{dt} = 27 \text{ mi/h}$

find $\frac{dr}{dt}$ when $t = 12 \text{ min} = \frac{12}{60} \text{ hr} = \frac{1}{5} \text{ hr}$

$$x^2 + y^2 = r^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

find x

$x = 27\left(\frac{1}{5}\right)$

$x = \frac{27}{5} = 5.4 \text{ mi}$

find y

$y = 32\left(\frac{1}{5}\right)$

$y = \frac{32}{5} = 6.4 \text{ mi}$

find r $\quad (5.4)^2 + (6.4)^2 = r^2$

$70.12 = r^2$

$r = \sqrt{70.12}$

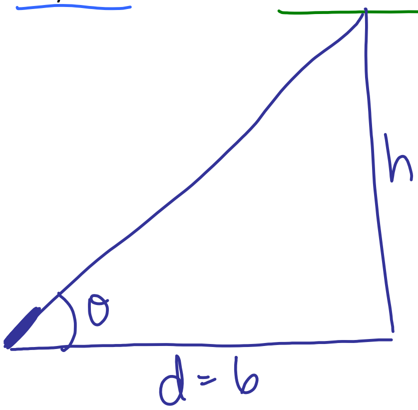
$$(5.4)27 + 6.4(32) = \sqrt{70.12}\,\frac{dr}{dt}$$

$$\frac{350.6}{\sqrt{70.12}} = dr/dt$$

$$dr/dt = 41.869 \text{ mi/hr}$$

Example #6

An observer watches a rocket launch using a telescope. The launching pad is 6km away. At a certain time the angle θ between the telescope and the ground is equal to $\frac{\pi}{3}$ and is changing at a rate of 0.9 rad/min. What is the rocket's velocity at that moment?

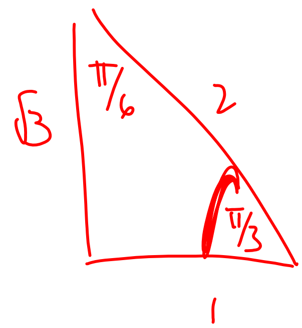


find velocity = dh/dt

h = height of rocket

d = distance

$\theta = \frac{\pi}{3}$ $\quad \frac{d\theta}{dt} = .9$



$\sec\frac{\pi}{3} = \frac{1}{\cos\pi/3} = \frac{1}{\frac{1}{2}}$

$$\tan\theta = \frac{h}{6}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{6}(1)\frac{dh}{dt}$$

$$\left(\sec\frac{\pi}{3}\right)^2(.9) = \frac{1}{6}\frac{dh}{dt}$$

$$6(2^2)(.9) = \frac{dh}{dt}$$

$$\begin{array}{r} 24 \\ \times .9 \\ \hline 21.6 \end{array}$$

$$21.6 \text{ Km/min} = \frac{dh}{dt}$$