4.1 Linear Approximation

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4.1 Linear Approximation

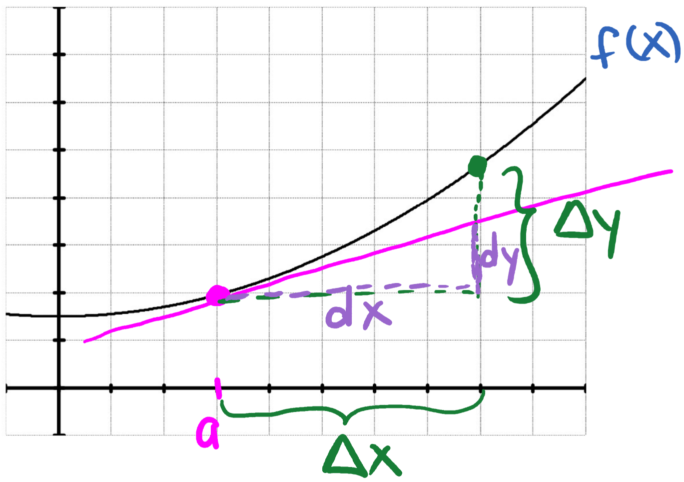
Linear Approximation: Tangent Line Approximation

Equation of a line: $y - y_1 = m(x - x_1)$ At the point (x_1, y_1) with a slope of m

Tangent Line:

point $(a, f(a))$ slope $f'(a)$

$y - f(a) = f'(a)(x-a)$



slope of tangent line
$m = \frac{dy}{dx} = f'(a)$

As $\Delta x \to 0$ $\Delta y \approx dy$ then the tangent line is a good approximation for f(x).

$\Delta x \to 0$ $dy \approx \Delta y = \Delta f$

$\frac{dy}{dx} = f'(a)$

$dx = \Delta x$

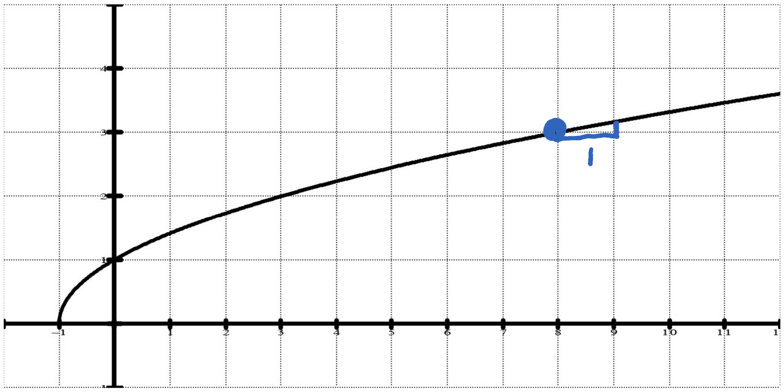
$dy = f'(a)\,dx$

as $\Delta x \to 0$
$dy \approx \Delta y$

$\Delta y = f'(a)\,\Delta x$

$\Delta f = f'(a)\,\Delta x$

1. Let $f(x) = \sqrt{1+x}$ use a=8 and $\Delta x = 1$



$f(x) = (1+x)^{\frac{1}{2}}$

a) Find an approximation for Δf

$\Delta f = f'(a)\Delta x$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} \cdot (1)$

$f'(8) = \frac{1}{2}(8+1)^{-\frac{1}{2}}$

$f'(8) = \frac{1}{2}(9)^{-\frac{1}{2}}$

$f'(8) = \frac{1}{2}\left(\frac{1}{3}\right)$

$f'(8) = \frac{1}{6}$

$\Delta f = f'(8)(1)$

$\Delta f = \frac{1}{6}(1)$

$\Delta f = \frac{1}{6}$

b) Find the actual change in f

$\Delta f = f(a+\Delta x) - f(a)$

$\Delta f = f(8+1) - f(8)$

$\Delta f = f(9) - f(8)$

$\Delta f = \sqrt{1+9} - \sqrt{1+8}$

$\Delta f = \sqrt{10} - \sqrt{9}$

$\Delta f = \sqrt{10} - 3$

$\Delta f = 0.1622776602$

c) Find the error $Error = |Actual\ \Delta f - Approximate\ \Delta f|$

$$Error = \left| 0.1622776602 - \frac{1}{6} \right|$$

$$Error = 0.004389006 5$$

d) Find the percentage error $Percentage\ Error = \left|\frac{Error}{Actual\ \Delta f}\right| 100\%$

$$\%\ error = \left| \frac{0.004389006 5}{0.1622776602} \right| 100\%$$

$$\%\ error = 2.7046\%$$

2. Actual $\Delta f = (16.5)^{\frac{1}{4}} - (16)^{\frac{1}{4}}$. Find an approximation for Δf and find the error.

$$\text{Actual } \Delta f = f(a + \Delta x) - f(a)$$

$$\text{Approximate } \Delta f = f'(a)\, \Delta x$$

$a = 16$

$\Delta x = 0.5$

$f(x) = x^{1/4}$

$f'(x) = \frac{1}{4} x^{-\frac{3}{4}}$

$f'(16) = \frac{1}{4}(16)^{-3/4}$

$f'(16) = \frac{1}{4}\left(\frac{1}{8}\right)$

$f'(16) = \frac{1}{32}$

$\Delta f = f'(16)(0.5)$

$\Delta f = \frac{1}{32} \cdot \frac{1}{2}$

$\Delta f = \frac{1}{64}$

$$Error = \left| (16.5)^{\frac{1}{4}} - 16^{\frac{1}{4}} - \frac{1}{64} \right|$$

$$Error = 1.7984 \times 10^{-4}$$

3. $f(x) = x^2$ Find the linearization of f(x) at the point (1,1)

$L(x) = f'(a)(x-a) + f(a)$

$f(x) = x^2$

$a = 1$

$f(1) = 1$

$f'(x) = 2x$

$L(x) = 2(1)(x-1) + 1$

$L(x) = 2(x-1) + 1$

$L(x) = 2x - 2 + 1$

$L(x) = 2x - 1$

4. Use linear approximation to find a value for $(27.03)^{\frac{1}{3}}$

Find L(x), then find L(27.03)

$a = 27$

$\Delta x = 0.03$

$f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(27) = \frac{1}{3}(27)^{-2/3}$

$f'(27) = \frac{1}{3}\left(\frac{1}{9}\right)$

$f'(27) = \frac{1}{27}$

$f(27) = 27^{1/3}$

$f(27) = 3$

$L(x) = f'(a)(x-a) + f(a)$

$L(x) = \frac{1}{27}(x-27) + 3$

$L(x) = \frac{1}{27}x - 1 + 3$

$L(x) = \frac{1}{27}x + 2$

$L(27.03) = \frac{1}{27}(27.03) + 2$

$L(27.03) = 3.00\overline{1}$