Note Title

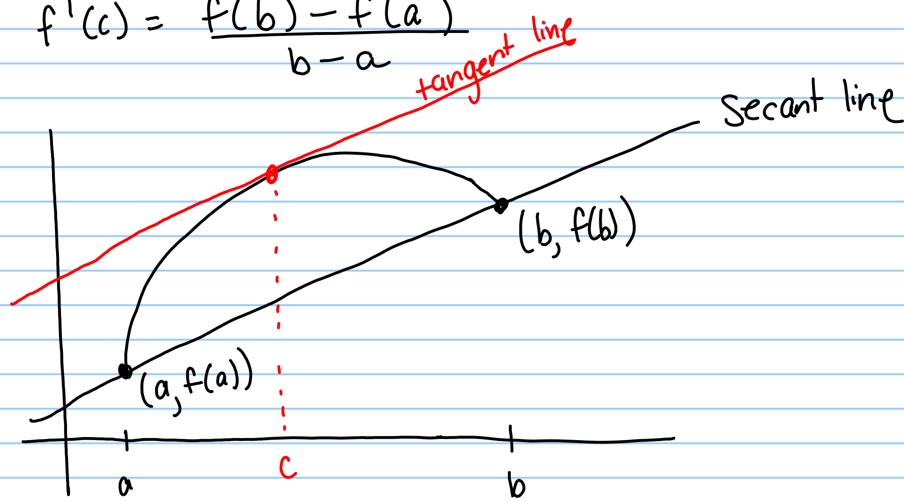
4.3 Mean Value Theorem and Monotonicity

11/18/2014

Mean Value Theorem (MVT)

$f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one "c" on (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



slope of secant $= \frac{f(b) - f(a)}{b - a}$ slope of tangent $f'(c)$

MVT implies slope of secant = slope of the tangent at $x = c$

average rate of change $[a, b]$ = instantaneous rate of change on (a, b) at point $x = c$

#1 Verify the Mean Value theorem for $f(x) = 5 - \frac{4}{x}$ on $(1, 4)$

$f(x)$ is continuous on $[1, 4]$

differentiable on $(1, 4)$

$f(4) = 5 - \frac{4}{4} = 4$ $f(x) = 5 - 4x^{-1}$

$f(1) = 5 - \frac{4}{1} = 1$

$f'(x) = 0 - 4(-1)x^{-2}$

$f'(x) = \frac{4}{x^2}$

$$\frac{f(4) - f(1)}{4 - 1} = f'(c)$$

$$\frac{4 - 1}{3} = \frac{4}{c^2}$$

$$\frac{3}{3} = \frac{4}{c^2}$$

$$c^2 = 4$$

$$c = \pm 2$$

$c = -2$ is not on the interval

$\therefore$ at $(2, f(2))$ slope of the tangent = slope of the secant

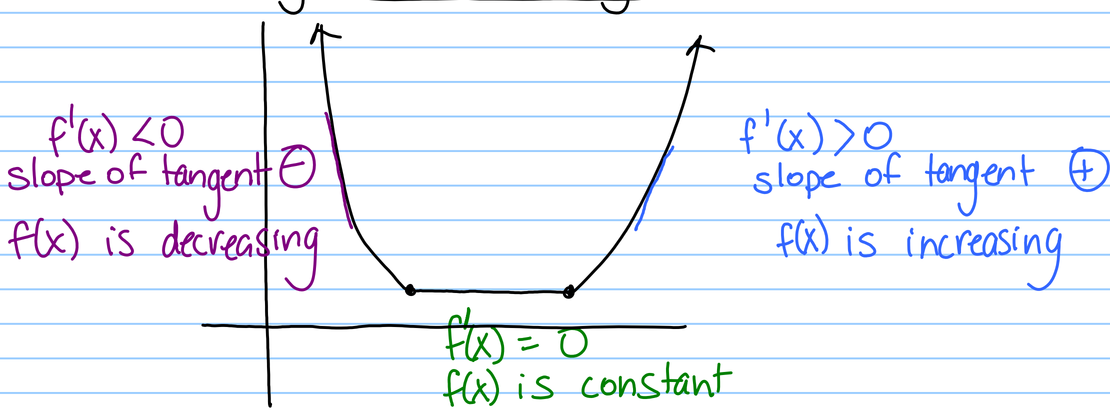
$(2, 3)$

$f(2) = 5 - \frac{4}{2}$

$= 5 - 2$

$= 3$

Increasing and Decreasing Behaviour of Functions

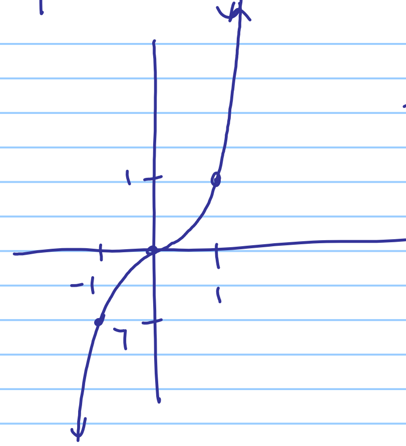


Increasing on (a,b) $\quad f(x_1) < f(x_2) \quad x_1 < x_2$

Decreasing on (a,b) $\quad f(x_1) > f(x_2) \quad x_1 < x_2$

$f(x)$ is monotonic on (a,b) if the function is always increasing or always decreasing.

eg $y = x^3$



$f'(x) > 0$
always increasing

#2 Find the interval where $f(x)$ is increasing and decreasing

$$f(x) = x^4 - 4x^3 - 8x^2 - 1$$

$$f'(x) = 4x^3 - 12x^2 - 16x$$
$$0 = 4x^3 - 12x^2 - 16x$$
$$0 = 4x(x^2 - 3x - 4)$$
$$0 = 4x(x-4)(x+1)$$

$x = 0 \quad x = 4 \quad x = -1$

Intervals	$(-\infty, -1)$	$(-1, 0)$	$(0, 4)$	$(4, \infty)$
test Value	$x = -2$	$x = -.5$	$x = 1$	$x = 5$
sign of $f'(x)$	$-$	$+$	$-$	$+$
	decreasing	increasing	decreasing	increasing

$f(x)$ is increasing on $(-1, 0)$ and $(4, \infty)$ as $f'(x) > 0$ or $f'(x)$ is positive

$f(x)$ is decreasing on $(-\infty, -1)$ and $(0, 4)$ as $f'(x)$ is negative or $f'(x) < 0$

First Derivative test for Critical Points!

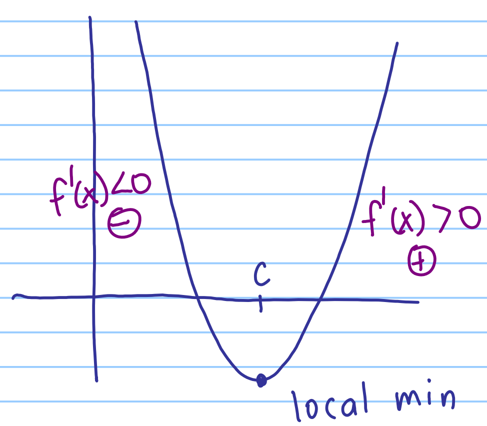
$f(x)$ is differentiable and c is a critical point of f.

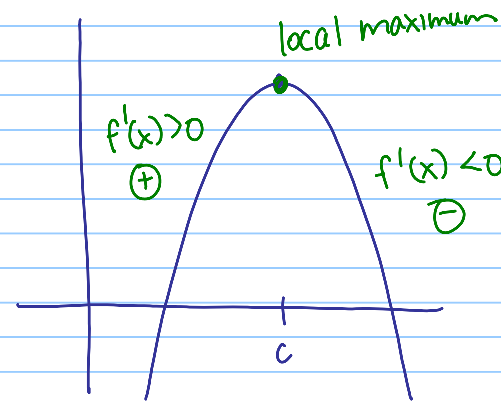
critical points when $f'(c) = 0$
or
$f'(c)$ undefined.

Then

$f'(x)$ changes from $\oplus$ to $\ominus$ at c then $f(c)$ is a local maximum.

$f'(x)$ changes from $\ominus$ to $\oplus$ at c then $f(c)$ is a local minimum.





#3 Find the relative extrema (local extrema) of

$$f(x) = \frac{x^4+1}{x^2}$$

$$f'(x) = \frac{x^2(4x^3) - (x^4+1)(2x)}{(x^2)^2}$$

$$f'(x) = \frac{4x^5 - 2x^5 - 2x}{x^4}$$

$$f'(x) = \frac{2x^5 - 2x}{x^4}$$

$$f'(x) = \frac{2x^4 - 2}{x^3}$$

$f'(x) = 0$ $\quad$ $f'(x)$ is undefined

$2x^4 - 2 = 0$ $\quad$ $x^3 = 0$

$x^4 - 1 = 0$ $\quad$ $x = 0$

$x = \pm 1$

critical #'s $x = -1$, $x = 0$, $x = 1$

$$f'(x) = \frac{2x^4 - 2}{x^3} = \frac{2(x^4-1)}{x^3}$$

intervals	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
test value	$x = -2$	$x = -.5$	$x = .5$	$x = 2$
sign of $f'(x)$	$-$	$+$	$-$	$+$
	decreasing	increasing	decreasing	increasing

$x = -1$ $\quad$ $f(-1) = \dfrac{(-1)^4 + 1}{(-1)^2} = 2$ $\quad$ $(-1, 2)$

Local minimum $f'(x)$ changes from $-$ to $+$

$x = 0$ $\quad$ $f(0) = \dfrac{0^4 + 1}{0^2}$ undefined $\quad$ No max No min

$x = 1$ $\quad$ $f(1) = \dfrac{1^4 + 1}{1^2} = 2$ $\quad$ $(1, 2)$

Local minimum $f'(x)$ changes from $-$ to $+$