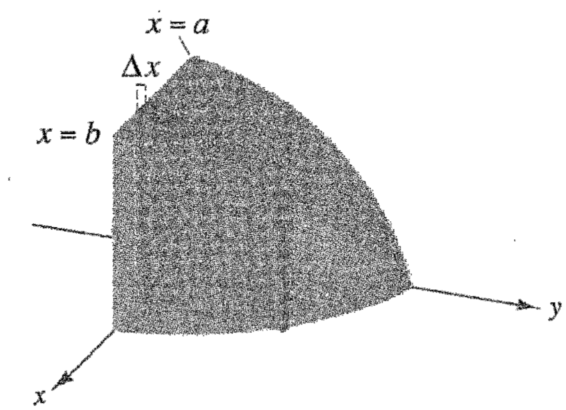
6.2 Setting Up Integrals: Volume, Density, Average Value

Volume as the integral of Cross-Sectional Area:

Let *A(x)* be the area of the cross-section that is perpendicular to the x-axis.

$Volume = \int_a^b A(x)dx$



A(x) or A(y)

Area square

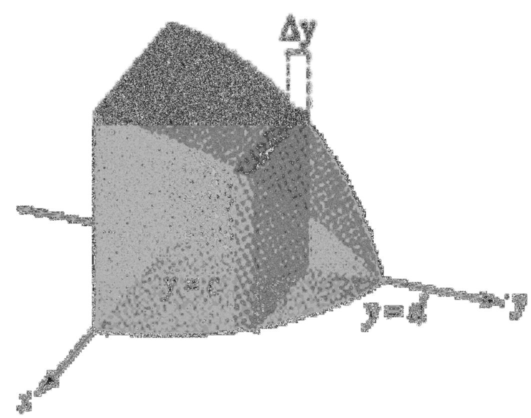
Area semicircle

Area triangle

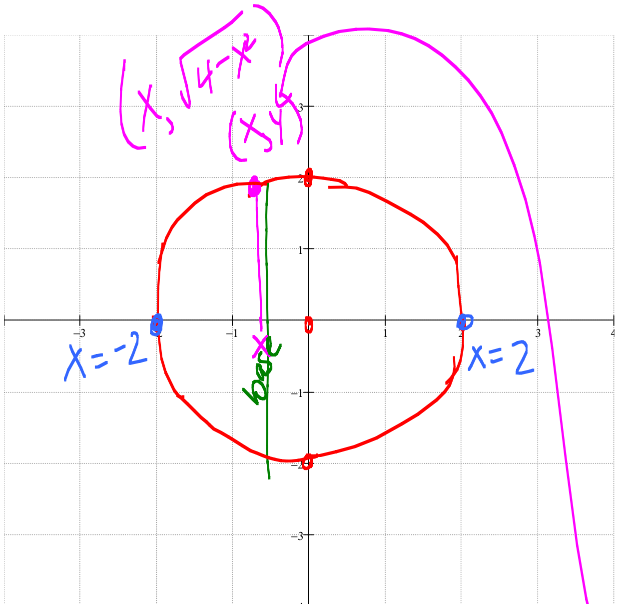
Area rectangle

Let *A(y)* be the area of the cross-section that is perpendicular to the y-axis.

$Volume = \int_c^d A(y)dy$



1. Find the volume of the solid whose base is bounded by the circle, $x^2 + y^2 = 4$, with the cross-sections of squares perpendicular to the x-axis.

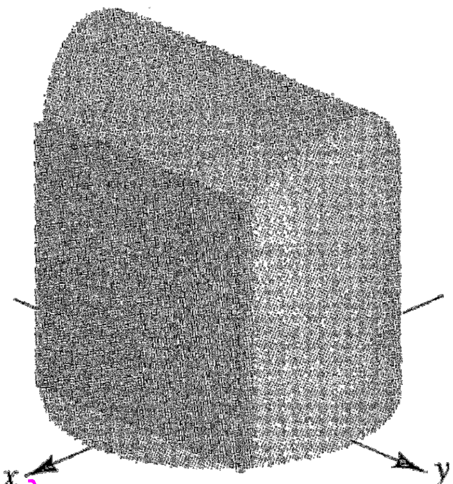


$x^2 + y^2 = r^2$

$r^2 = 4$

radius = 2

center $(0,0)$



$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

$y = \sqrt{4-x^2}$

base $= 2\sqrt{4-x^2}$

$A(x)$ = area square

$A(x) = (\text{base})(\text{base})$

$A(x) = 2\sqrt{4-x^2} \cdot 2\sqrt{4-x^2}$

$A(x) = 4(4-x^2)$

$$V = \int_{-2}^{2} 4(4-x^2)\,dx$$

$$V = 4\left[4x - \frac{1}{3}x^3\right]_{-2}^{2}$$

$$V = 4\left[8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right]$$

$$V = 4\left[8 - \frac{8}{3} + 8 - \frac{8}{3}\right]$$

$$V = 4\left[\frac{32}{3}\right]$$

$$V = \frac{128}{3}$$

Radial Density Function: *p(r)* is a function that depends on the distance r from the center of a city. It determines the population density (people per square kilometer or mile)

The **actual population *P*** within a radius *R* is given by

$$P = 2\pi \int_0^R r p(r) dr$$

2. A national park [redacted] has a high density of gorillas. Suppose that the radial density function is [redacted] gorillas per square kilometer, where *r* is the distance from a large grassy cleari[redacted] and water. Calculate the number of gorillas within a 5 km radius of the clearing.

$$P = 2\pi \int_0^5 r \cdot 52(1+r^2)^{-2} dr$$

$$P = 104\pi \int_0^5 r(1+r^2)^{-2} dr$$

$u = 1+r^2$

$\frac{du}{dr} = 2r$

$\frac{du}{2} = r\,dr$

$$P = 104\pi \int_{r=0}^{r=5} u^{-2} \cdot \frac{du}{2}$$

$$P = 52\pi \left[\frac{u^{-1}}{-1}\right]_{r=0}^{r=5}$$

$$P = 52\pi \left[\frac{-1}{1+r^2}\right]_0^5$$

$$P = 52\pi \left[\frac{-1}{1+5^2} - \frac{-1}{1+0^2}\right]$$

$$= 52\pi \left[\frac{-1}{26} + 1\right]$$

$$= 52\pi \left[\frac{25}{26}\right] = 50\pi \approx 157 \text{ gorillas}$$

Average Value: The average value of an integrable function f(x) on [a,b] is the quantity

$$Average\ value = \frac{1}{b-a}\int_a^b f(x)dx$$

$$(\text{Average value})(b-a) = \int_a^b f(x)dx$$

$$(\text{height})\ (\text{width}) = \text{area under curve}$$

Mean Value theorem for integrals: If f(x) is continuous on [a,b], then there exists a value c on [a,b] such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

3. Find the average value of $f(x) = 3x^2 - 2x$ on the interval [1,4].

$$\text{Average Value} = \frac{1}{4-1}\int_1^4 (3x^2-2x)dx$$

$$AV = \frac{1}{3}\Big[x^3 - x^2\Big]_1^4$$

$$\text{Average Value} = \frac{1}{3}\left[4^3 - 4^2 - (1^3 - 1^2)\right]$$

$$= \frac{1}{3}[64-16]$$

$$= \frac{48}{3} = 16$$

4. Let *M* be the average value of $f(x) = 2x^2$ on [0,2]. Find a value *c* such that $f(c) = M$

$$M = \frac{1}{2-0}\int_0^2 2x^2\,dx$$

$$M = \frac{1}{2}\left[\frac{2}{3}x^3\right]_0^2$$

$$M = \frac{1}{2}\left[\frac{2}{3}(2)^3 - \frac{2}{3}(0)\right]$$

$$M = \frac{1}{2}\left[\frac{16}{3}\right]$$

$M = \frac{8}{3}$ = height of rectangle

$$f(c) = M$$

$$2c^2 = \frac{8}{3}$$

$$c^2 = \frac{4}{3}$$

$$c = \pm\frac{2}{\sqrt{3}}$$

only $c = \frac{2}{\sqrt{3}}$ is on the interval $[0,2]$