7.3 Part 1

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AP Calculus 12

7.3 Part 1 Logarithms and Derivatives

1. Use the definition of a derivative to find $f'(x)$ if $f(x) = b^x$

$$f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{b^x (b^h - 1)}{h}$$

$$f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

$$f'(x) = b^x \cdot \ln b$$

$$\boxed{\frac{d}{dx} b^u = b^u \cdot \ln b \cdot \frac{du}{dx}}$$

2. Find the derivative $f(x) = 2^{3x}$

$$f'(x) = 2^{3x} \cdot \ln 2 \cdot \frac{d}{dx} 3x$$

$$f'(x) = 2^{3x} \cdot \ln 2 \cdot 3$$

$$f'(x) = 2^{3x} (\ln 8)$$

$\ln 2 \cdot 3$

$3 \ln 2$

$\ln 2^3$

$\ln 8$

$y - y_1 = m(x - x_1)$ or $y - f(a) = f'(a)(x - a)$

$y - y_1 = m(x - x_1)$ or $y - f(a) = f'(a)(x - a)$

3. Find the equation of the tangent line of $f(x) = 2^{x^2-x}$ at $x = -1$

point
(x_1, y_1)
$(a, f(a))$

$f(-1) = 2^{(-1)^2-(-1)}$
$f(-1) = 2^2$
$f(-1) = 4$

$(-1, 4)$

slope
m
$f'(a)$

$f'(x) = 2^{x^2-x} \cdot \ln 2 \cdot (2x-1)$
$f'(-1) = 2^{(-1)^2-(-1)} \cdot \ln 2 \cdot (2(-1)-1)$
$f'(-1) = 4\ln 2(-3)$
$f'(-1) = -12\ln 2$

$y - 4 = -12\ln 2(x - (-1))$
$y - 4 = -12\ln 2(x + 1)$
$y = -12\ln 2(x+1) + 4$

Derivative of the natural log

Let $g(x) = \ln x$ and $f(x) = e^x$

$f(x)$ *and* $g(x)$ are inverses:

$f(g(x)) = x$

$\frac{d}{dx} f(g(x)) = \frac{d}{dx} x$

$f'(g(x)) \cdot g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$ *

$f(x) = e^x$
$f'(x) = e^x = f(x)$

$g'(x) = \frac{1}{f(g(x))}$

$g'(x) = \frac{1}{x}$

$\frac{d}{dx} \ln x = \frac{1}{x}$

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}}$$

4. $f(x) = \ln(x^2+1)$ $find\ f'(x)$

$$f'(x) = \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) \quad *$$

$$f'(x) = \frac{1}{x^2+1} \cdot 2x \quad \checkmark$$

$$f'(x) = \frac{2x}{x^2+1}$$

5. $y = \ln(\boxed{\ln x})$ Find y'

$$y' = \frac{1}{\ln x} \cdot \frac{d}{dx}\ln x$$

$$y' = \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$y' = \frac{1}{x\ln x}$$

6. $f(x) = \ln\sqrt{3x+1}$ find f'(x)

$$f'(x) = \frac{1}{\sqrt{3x+1}} \cdot \frac{1}{2} \cdot (3x+1)^{-\frac{1}{2}} \cdot 3$$

$$f'(x) = \frac{3}{\sqrt{3x+1} \cdot 2\,(3x+1)^{1/2}}$$

$$f'(x) = \frac{3}{2(3x+1)}$$

$$f(x) = \ln(3x+1)^{1/2}$$

$$f(x) = \frac{1}{2}\ln(3x+1)$$

$$f'(x) = \frac{1}{2(3x+1)} \cdot 3$$

$$f'(x) = \frac{3}{2(3x+1)}$$

Derivative of logs

Let $f(x) = \log x$ or $f(x) = \log_{10} x$

We know how to take derivatives of natural logs so use the change of base formula to change f(x) into base e or $\ln x$. Then find the derivative of f(x).

change of base $\log_c x = \dfrac{\log_b x}{\log_b c} = \dfrac{\log_e x}{\log_e c} = \dfrac{\ln x}{\ln c}$

$f(x) = \log_{10} x = \dfrac{\ln x}{\ln 10} = \ln x \cdot \dfrac{1}{\ln 10}$

$f'(x) = \dfrac{1}{\ln 10} \cdot \dfrac{d}{dx} \ln x$

$f'(x) = \dfrac{1}{\ln 10} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln 10}$

$$\boxed{\frac{d}{dx} \log_b u = \frac{1}{u \ln b} \cdot \frac{du}{dx}}$$

7. $g(x) = \log_3 (3x+1)^4$ Find g'(x)

$g'(x) = \dfrac{1}{(3x+1)^{\cancel{4}} \cdot \ln 3} \cdot 4\cancel{(3x+1)}^{3} \cdot 3$

$g'(x) = \dfrac{12}{(3x+1)\ln 3}$

$g(x) = 4 \log_3 (3x+1)$

$g'(x) = 4 \cdot \dfrac{1}{(3x+1)\ln 3} \cdot 3$

$g'(x) = \dfrac{12}{(3x+1)\ln 3}$

8. $y = \ln \frac{x}{\sqrt{x+1}}$ find y'

$$y' = \frac{1}{\frac{x}{\sqrt{x+1}}} \cdot \left[\frac{\sqrt{x+1} \cdot 1 - x \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}} \cdot 1}{\left(\sqrt{x+1}\right)^2} \right]$$

$$y' = \frac{\sqrt{x+1}}{x} \left[\frac{\sqrt{x+1} - \frac{x}{2\sqrt{x+1}}}{x+1} \right]$$

$$y' = \frac{x+1 - \frac{x}{2}}{x(x+1)}$$

$$y' = \frac{\frac{2x+2}{2} - \frac{x}{2}}{x(x+1)}$$

$$y' = \frac{\frac{x+2}{2}}{\frac{x(x+1)}{1}} = \frac{x+2}{2} \cdot \frac{1}{x(x+1)} = \frac{x+2}{2x(x+1)}$$

$$y = \ln x - \ln\sqrt{x+1}$$

$$y = \ln x - \ln(x+1)^{1/2}$$

$$y = \ln x - \frac{1}{2}\ln(x+1)$$

$$y' = \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x+1}$$

$$y' = \frac{2(x+1)}{x \cdot 2(x+1)} - \frac{x}{x \cdot 2(x+1)}$$

$$y' = \frac{x+2}{2x(x+1)}$$

Logarithmic Differentiation: Apply a log or ln to both sides of the function. Use log/ln rules to simplify the function. Take the derivative of the simplified function.

~~$y = x^2$~~ ~~$y' = 2x$~~

~~$y = 2^x$~~ ~~$y' = 2^x \ln 2$~~

9. $y = x^{x^3}$ find y'

$$\ln y = \ln x^{x^3}$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} \ln x^{x^3}$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} x^3 \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3x^2 \cdot \ln x + x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}}$$

$$\frac{dy}{dx} = y\left[3x^2 \ln x + x^2\right]$$

$$\frac{dy}{dx} = x^{x^3}\left[x^2(3\ln x + 1)\right]$$

$$\frac{dy}{dx} = x^{x^3} \cdot x^2 \, (3\ln x + 1)$$

$$\frac{dy}{dx} = x^{x^3+2}(3\ln x + 1)$$

10. $y = \frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}$ *find* y'

$$\ln y = \ln \frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}$$

$$\ln y = \ln e^x\sqrt{x^2+1} - \ln (x^2+2)^3$$

$$\ln y = \ln e^x + \ln(x^2+1)^{1/2} - \ln(x^2+2)^3$$

$$\ln y = x + \tfrac{1}{2}\ln(x^2+1) - 3\ln(x^2+2)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{1}{\cancel{2}} \cdot \frac{1}{x^2+1} \cdot \cancel{2}x - 3 \cdot \frac{1}{x^2+2} \cdot 2x$$

$$\frac{dy}{dx} = y\left[1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right]$$

$$\frac{dy}{dx} = \frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}\left[1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right]$$