7.4 7.5 New

Monday, March 4, 2024 1:19 PM

AP Calculus 12

7.4 and 7.5 Exponential Growth and Decay

Review Math 12 Exponential Functions:

Growth and Decay Problems	Compound Interest
$A = A_0(c)^{\frac{t}{T}}$ A = final amount A_0= initial amount c = rate of change • Doubles c = 2 • Half-life c = ½ $c = 100\% + 2\%$ • Growth by 2% c = 1 + 0.02 = $\boxed{1.02}$ • Depreciates by 3% c = 1 – 0.03 = 0.97 t = time T = how often the rate makes a change	$A = P\left(1+\frac{r}{n}\right)^{nt}$ A = final amount P = initial amount (Principle) r = rate (as a decimal) t = time in years n = number of times compounded per year • Compounded annually n = 1 • Compounded quarterly n = 4 • Compounded monthly n = 12 • Compounded weekly n = 52

1. Given $y = 8e^{3t}$ find y'

$$y' = \underbrace{8e^{3t}}_{y} \cdot 3$$

$$y' = y \cdot 3$$

$$y' = 3y$$

If $y(t)$ is a differentiable function satisfying the differential equation $y' = ky$

then $y(t) = P_0e^{kt}$, where $P_0 = y(0)$ – Initial Population

$$y'(t) = \underbrace{P_0 e^{kt}}_{y} \cdot K$$

$$y'(t) = y \cdot K$$

$$y'(t) = Ky$$

2. A bacterial culture starts with 2000 bacteria and after 3h the estimated count is 10000. Assume the ~~bactia~~ bacteria is growing exponentially

a) Using the equation $\boxed{y(t) = P_0 e^{kt}}$ and find the exponential equation that models the situation

$P_0 = 2000$

$y(t) = 2000e^{Kt}$ $\quad y(3) = 10000$

$10000 = 2000e^{K(3)}$

$5 = e^{3K}$

$\ln 5 = \ln e^{3K}$

$\ln 5 = 3K \ln e$ $\quad \ln e = 1$

$\ln 5 = 3K(1)$ $\quad (3^2)^5 = 3^{10}$

$\frac{\ln 5}{3} = K$

$y(t) = 2000e^{\frac{\ln 5}{3}t}$

$y(t) = 2000(e^{\ln 5})^{t/3}$ $\quad c^{\log_c x} = x$

$e^{\ln 5} = 5$

$y(t) = 2000(5)^{t/3}$

b) Find the rate of growth after 2h

derivative

$y'(t) = 2000 \cdot 5^{t/3} \cdot \ln 5 \cdot \frac{1}{3}$

$y'(t) = \frac{\ln 5}{3} \cdot \underbrace{2000(5)^{t/3}}$

$y'(t) = K \cdot y$

$y'(2) = \frac{\ln 5}{3} \cdot 2000(5)^{2/3}$

$y'(2) = 3137.35$ bacteria/hrs

c) When will the bacteria population reach 18000?

$y = 18000$ find t

$18000 = 2000(5)^{t/3}$

$9 = 5^{t/3}$

$\ln 9 = \ln 5^{t/3}$

$\ln 9 = \frac{t}{3}\ln 5$

$\frac{3\ln 9}{\ln 5} = t$

$t = 4.1$ hrs.

3. The half-life of Polonium is 140d and a sample of this element has a mass of 300mg.

a) Using the equation $y(t) = P_0 e^{kt}$ and find the exponential equation that models the situation

$P_0 = 300$

$y(t) = 300e^{kt}$ $\quad y(140) = 150$

$150 = 300e^{k(140)}$

$\frac{1}{2} = e^{140k}$

$\ln\frac{1}{2} = \ln e^{140k}$

$\ln\frac{1}{2} = 140k \ln e$

$\boxed{\frac{\ln\frac{1}{2}}{140} = k}$

$y(t) = 300e^{\frac{\ln\frac{1}{2}}{140}t}$

$y(t) = 300\left(e^{\ln\frac{1}{2}}\right)^{t/140}$

$y(t) = 300\left(\frac{1}{2}\right)^{t/140}$

b) Find the rate of decrease of the mass after 50d

derivative

$y' = ky$

$y' = \frac{\ln\frac{1}{2}}{140} \cdot 300\left(\frac{1}{2}\right)^{t/140}$

$y'(50) = \frac{\ln\frac{1}{2}}{140} \cdot 300\left(\frac{1}{2}\right)^{50/140}$

$y'(50) = -1.159$ mg/day

c) How long will the sample take to decay to a mass of 200mg

$y = 200$ find t

$200 = 300\left(\frac{1}{2}\right)^{t/140}$

$\frac{2}{3} = \left(\frac{1}{2}\right)^{t/140}$

$\ln\frac{2}{3} = \ln\left(\frac{1}{2}\right)^{t/140}$

$\ln\frac{2}{3} = \frac{t}{140}\ln\frac{1}{2}$

$\frac{140\ln\frac{2}{3}}{\ln\frac{1}{2}} = t$

$t = 81.89$ days

The equation $\boxed{y(t) = P_0 e^{kt}}$ is written as $\boxed{y = Pe^{rt},}$ * where r is the rate of growth/decay.

- population problems where there is continuous growth
- investment problems where interest is $\boxed{\text{compounded continuously}}$

4. A sum of $1000 is invested at an interest rate of 12% per annum. Find the amount in the account after 10 years if interest is compounded:

a) monthly

$P = 1000$
$r = 12\% = 0.12$
$t = 10$
$n = 12$

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 1000\left(1 + \frac{0.12}{12}\right)^{12(10)}$

$A = \$3300.39$

b) compounded continuously.

$P = 1000$
$r = 12\% = 0.12$
$t = 10$

$y = Pe^{rt}$

$y = 1000e^{0.12(10)}$

$y = \$3320.12$