7.7 L'Hopital

Thursday, March 9, 2023 8:59 AM

7.7 L'Hopital

1. Direct substitution
   If you get the indeterminate form using direct substitution then try one of the other techniques $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, 1^∞
2. Factor and simplify
3. Divide all terms by the highest power in the denominator
4. Multiply by the conjugate and simplify $\sqrt{}$
5. Use a trig identity and simplify

1. $\lim\limits_{x \to 3} \frac{x^2-9}{x-3}$

$$\frac{3^2-9}{3-3} = \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$$

$$= \lim_{x \to 3} x+3$$

$$= 3+3$$

$$= 6$$

2. $\lim\limits_{x \to \infty} \frac{5x^3+3x+2}{3x^3+x^2+2x}$

$$\frac{\infty + \infty + 2}{\infty + \infty + \infty} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{\frac{5x^3}{x^3} + \frac{3x}{x^3} + \frac{2}{x^3}}{\frac{3x^3}{x^3} + \frac{x^2}{x^3} + \frac{2x}{x^3}}$$

$$\lim_{x \to \infty} \frac{5 + \frac{3}{x^2} + \frac{2}{x^3}}{3 + \frac{1}{x} + \frac{2}{x^2}}$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$= \frac{5+0+0}{3+0+0}$$

$$= 5/3$$

need a quotient

L'Hopitals Rule

If $\lim_{x\to c} \boxed{\frac{f(x)}{g(x)}}$ produces the indeterminate form and *f(x)* and *g(x)* are differentiable functions on (a,b) then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

3. $\lim_{x\to 0} \frac{e^{2x}-1}{x}$

Direct sub

$$\frac{e^0-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(e^{2x}-1)}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to 0} \frac{e^{2x}\cdot 2 - 0}{1}$$

$$= \lim_{x\to 0} 2e^{2x}$$

$$= 2e^{2(0)} = 2$$

4. $\lim_{x\to -\infty} \frac{x^2}{e^{-x}}$

direct sub

$$\frac{(-\infty)^2}{e^{\infty}} = \frac{\infty}{\infty}$$

$$= \lim_{x\to -\infty} \frac{\frac{d}{dx} x^2}{\frac{d}{dx} e^{-x}}$$

$$= \lim_{x\to -\infty} \frac{2x}{e^{-x}\cdot(-1)}$$

$$= \lim_{x\to -\infty} \frac{2x}{-e^{-x}}$$

direct sub

$$\frac{2(-\infty)}{-e^{\infty}} = \frac{-\infty}{-\infty}$$

$$= \lim_{x\to -\infty} \frac{\frac{d}{dx} 2x}{\frac{d}{dx} -e^{-x}}$$

$$= \lim_{x\to -\infty} \frac{2}{e^{-x}}$$

$$= \frac{2}{e^{\infty}}$$

$$= \frac{2}{\infty}$$

$$= 0$$

direct sub
$e^{-\infty} \cdot \sqrt{\infty}$
$0 \cdot \infty$

5. $\lim_{x\to\infty} e^{-x}\sqrt{x}$

$= \lim_{x\to\infty} \frac{x^{1/2}}{e^x}$

$= \lim_{x\to\infty} \frac{\frac{d}{dx} x^{1/2}}{\frac{d}{dx} e^x}$

$= \lim_{x\to\infty} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x}$

$= \lim_{x\to\infty} \frac{1}{2x^{1/2} \cdot e^x}$

$= \frac{1}{2\sqrt{\infty} \cdot e^{\infty}} = \frac{1}{\infty} = 0$

6. $\lim_{x\to 0} x^x$ direct sub 0^0

$y = \lim_{x\to 0} x^x$

$\ln y = \ln\left[\lim_{x\to 0} x^x\right]$

$\ln y = \lim_{x\to 0} \left(\ln x^x\right)$

$\ln y = \lim_{x\to 0} x \ln x$ make a quotient

$\ln y = \lim_{x\to 0} \frac{\ln x}{1/x}$

$\ln y = \lim_{x\to 0} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \frac{1}{x}}$

$\ln y = \lim_{x\to 0} \frac{1/x}{-1x^{-2}}$

$\ln y = \lim_{x\to 0} \frac{1}{\cancel{x}} \cdot -x^{\cancel{2}}$

$\ln y = \lim_{x\to 0} -x$

$\ln y = 0$

$\log_e y = 0$

$e^0 = y$ Exponential form

$1 = y$

$\lim_{x\to 0} x^x = 1$

Special Limit

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$x=5$

7. $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$

$= e^5$

$x=4$

8. $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{5n}$

$= \lim_{n\to\infty}\left[\left(1+\frac{4}{n}\right)^n\right]^5$

$= \left[\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n\right]^5$

$= \left[e^4\right]^5$

$= e^{20}$

$\lim_{x\to h}(f(x))^n = \left[\lim_{x\to h} f(x)\right]^n$

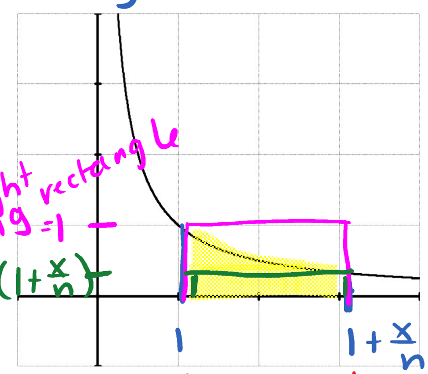
Proof of $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ using the squeeze theorem:

$$y=\frac{1}{t}=f(t)$$

$$\int_1^{1+\frac{x}{n}} \frac{1}{t}\,dt = \ln|t| \Big|_1^{1+\frac{x}{n}}$$

$$=\ln\left(1+\frac{x}{n}\right)-\ln(1)$$

$$=\ln\left(1+\frac{x}{n}\right)$$



height rectangle lg $=1$

$$\frac{1}{1+\frac{x}{n}} = f\left(1+\frac{x}{n}\right)$$

$$\frac{1}{\frac{n+x}{n}}$$

$$\frac{n}{n+x} = \text{height small rectangle}$$

$$\underbrace{\qquad}_{\text{width} = \frac{x}{n}}$$

$$\text{Small Area} \le \text{actual} \le \text{large area}$$

$$\frac{x}{\cancel{n}}\cdot\frac{\cancel{n}}{n+x} \le \ln\left(1+\frac{x}{n}\right) \le \frac{x}{n}\cdot 1$$

$$\frac{x}{n+x} \le \ln\left(1+\frac{x}{n}\right) \le \frac{x}{n}$$

$$\frac{nx}{n+x} \le n\ln\left(1+\frac{x}{n}\right) \le \cancel{n}\frac{x}{\cancel{n}}$$

$$\frac{nx}{n+x} \le \ln\left(1+\frac{x}{n}\right)^n \le x$$

$$\lim_{n\to\infty}\frac{nx}{n+x} \le \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n \le \lim_{n\to\infty} x$$

$$\lim_{n\to\infty}\frac{\frac{d}{dn}\,nx}{\frac{d}{dn}\,n+x} \le \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n \le x$$

$$\lim_{n\to\infty}\frac{1x}{1+0} \le \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n \le x$$

$$x \le \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n \le x$$

$$\therefore \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n = x$$

$n \to \infty$

$$\ln\left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right] = x$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$\log_e a = b$$

$$e^b = a$$