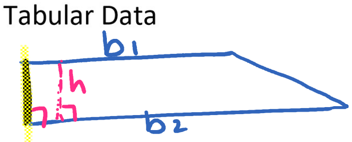
AP Calculus 12

8.8 Numerical Integration and Tabular Data

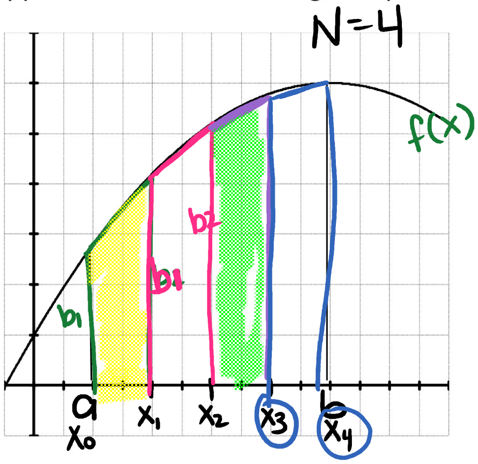
Trapezoidal Rule



$$A = \frac{(b_1+b_2)h}{2}$$

Approximate the area using 4 trapezoids

$N=4$



Δx = width

$$\Delta x = \frac{b-a}{N}$$

N = # of trapezoids

$$A = \frac{(f(x_0)+f(x_1))\Delta x}{2} + \frac{(f(x_1)+f(x_2))\Delta x}{2} + \frac{(f(x_2)+f(x_3))\Delta x}{2} + \frac{(f(x_3)+f(x_4))\Delta x}{2}$$

$$A = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right] \quad N=4$$

$$\int_a^b f(x)dx = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)]$$

$$\boxed{x_j = a + j\Delta x} \qquad \Delta x = \frac{b-a}{N}$$

j is a counter

Simpson's Rule (Not tested in AP but you may want this rule next year)

For N even $\qquad \Delta x = \frac{b-a}{N}$

$$\int_a^b f(x)dx = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{N-3}) + 2f(x_{N-2}) + 4f(x_{N-1}) + f(x_N)]$$

$$x_j = a + j\Delta x$$

1. Approximate $\int_0^{\pi} \sin x \; dx$ using the trapezoidal rule with 4 trapezoids (N=4)

$x_j = a + j\Delta x$

$\Delta x = \frac{b-a}{N}$ $\quad \Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$

$x_0 = 0 + 0\left(\frac{\pi}{4}\right) = 0$

$x_1 = 0 + 1\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$

$x_2 = 0 + 2\left(\frac{\pi}{4}\right) = \pi/2$

$x_3 = 0 + 3\left(\pi/4\right) = \frac{3\pi}{4}$

$x_4 = 0 + 4\left(\pi/4\right) = \pi$

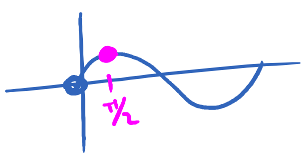
$$\int_0^{\pi} \sin x\,dx = \frac{\frac{\pi}{4}}{2}\left[f(0) + 2f\left(\tfrac{\pi}{4}\right) + 2f\left(\tfrac{\pi}{2}\right) + 2f\left(\tfrac{3\pi}{4}\right) + f(\pi)\right]$$

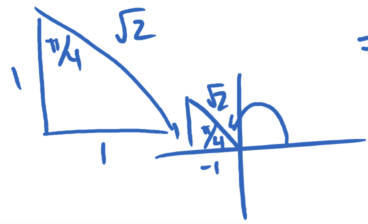
$$= \frac{\pi}{8}\left[\sin 0 + 2\sin\tfrac{\pi}{4} + 2\sin\tfrac{\pi}{2} + 2\sin\tfrac{3\pi}{4} + \sin\pi\right]$$

$$= \frac{\pi}{8}\left[0 + 2\left(\tfrac{1}{\sqrt{2}}\right) + 2(1) + 2\left(\tfrac{1}{\sqrt{2}}\right) + 0\right]$$

$$= \frac{\pi}{8}\left[\frac{4}{\sqrt{2}} + 2\right]$$

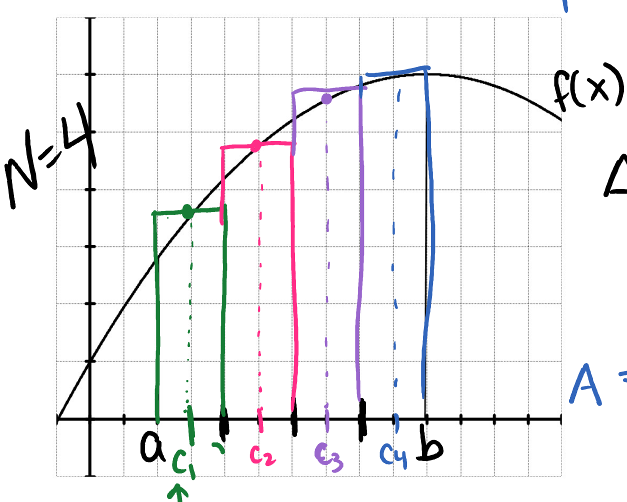
$$= 1.896$$





$$\int_0^{\pi} \sin x\,dx$$
$$= -\cos x \Big|_0^{\pi}$$
$$= -(-1) - (-1)$$
$$= 2$$

Midpoint Rule



$\Delta x = \frac{b-a}{N}$ $\quad N = \#$ of Recangles

$A = \Delta x \cdot f(c_1) + \Delta x \cdot f(c_2) + \Delta x f(c_3) + \Delta x f(c_4)$

$C_1 = a + \left(1 - \frac{1}{2}\right)\Delta x$

$C_1 = a + \frac{1}{2}\Delta x$

$$\int_a^b f(x)dx = \Delta x[f(c_1) + f(c_2) + \cdots + f(c_{N-1}) + f(c_N)]$$

$$c_j = a + \left(j - \tfrac{1}{2}\right)\Delta x \qquad \Delta x = \frac{b-a}{N}$$

2. Approximate $\int_1^2 \sqrt{x^4+1}\,dx$ using the midpoint rule with 5 rectangles (N=5)

$c_j = a + (j - \frac{1}{2})\Delta x$

$\Delta x = \frac{2-1}{5} = \frac{1}{5}$

$c_1 = 1 + (1 - \frac{1}{2})\frac{1}{5} = 1 + \frac{1}{10} = \frac{11}{10}$

$c_2 = 1 + (2 - \frac{1}{2})\frac{1}{5} = 1 + \frac{3}{10} = \frac{13}{10}$

$c_3 = 1 + (3 - \frac{1}{2})(\frac{1}{5}) = 1 + \frac{5}{10} = \frac{15}{10}$

$c_4 = 1 + (4 - \frac{1}{2})(\frac{1}{5}) = 1 + \frac{7}{10} = \frac{17}{10}$

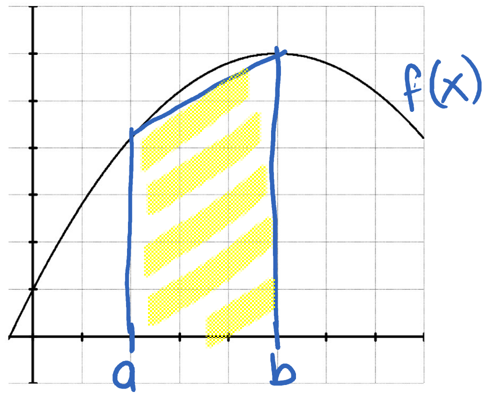
$f(x) = \sqrt{x^4+1}$

$c_5 = 1 + (5 - \frac{1}{2})(\frac{1}{5}) = 1 + \frac{9}{10} = \frac{19}{10}$

$$\int_1^2 \sqrt{x^4+1}\,dx = \frac{1}{5}\left[f(\tfrac{11}{10}) + f(\tfrac{13}{10}) + f(\tfrac{15}{10}) + f(\tfrac{17}{10}) + f(\tfrac{19}{10})\right]$$

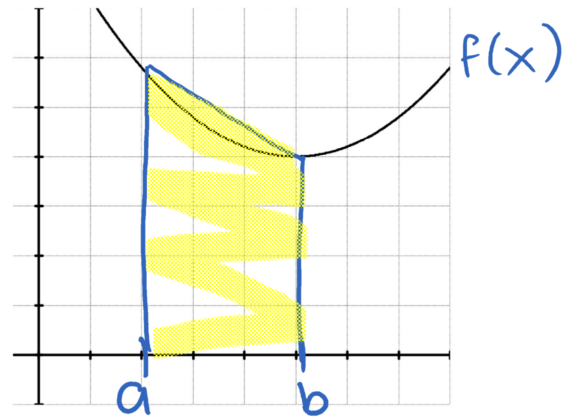
$$= \frac{1}{5}\left[1.570 + 1.964 + 2.462 + 3.058 + 3.746\right]$$

$$= 2.56$$



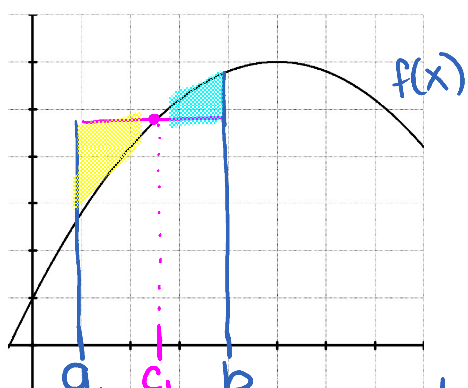
f(x) is concave down

Trapezoidal
Under estimate



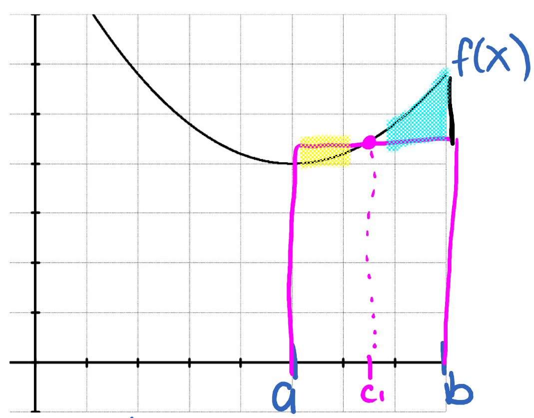
f(x) is concave up

Trapezoidal
Over estimate



f(x) is concave down

Midpoint is an
over estimate



f(x) is concave up

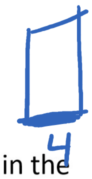
Midpoint is an
underestimate

Tabular Data (Information given in a chart)

3. Let $y(t)$ represent the population of a town over a 20 year period, where y is a differentiable function of t. The table below shows the population recorded at selected times.

12

$t(years)$	0	4	10	13	20
$y(t)(people)$	2500	2724	3108	3697	4283



a) Use the data from the table to find an approximation for $y'(12)$, and explain the meaning of $y'(12)$ in terms of the population of the town.

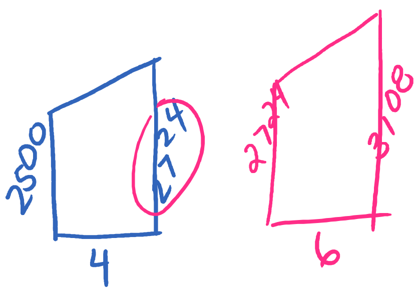
y' derivative
rate
slope

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$$y'(12) = \frac{y(13) - y(10)}{13 - 10}$$

$$= \frac{3697 - 3108}{3}$$

$$= 196.333 \text{ people/year}$$



b) Use data from the table and a trapezoidal approximation with four subintervals to approximate the average population of the town over the 20-year period.

$N = 4$

$$\text{Average Pop} = \text{Ave Value} = \frac{1}{b-a}\int_a^b y(t)\,dt$$

$$\int_0^{20} y(t)\,dt = \frac{(2500+2724)4}{2} + \frac{(2724+3108)6}{2} + \frac{(3108+3697)3}{2} + \frac{(3697+4283)7}{2}$$

$$= 66081.5$$

$$\text{Av Value} = \frac{1}{20-0}(66081.5)$$

$$= 3304.075 \text{ people}$$

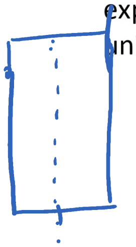
4. The rate at which water flows into a tank, in gallons per hour, is given by a positive continuous function R of time t. The table below shows the rate at selected values of t for a 12-hour period.

(4 — 8)

$t(hrs)$	0	2	4	6	8	10	12
$R(t)$ (gal/hr)	12.5	$\boxed{13.4}$	13.9	14.3	14.6	14.8	14.7

(0 — 4) (8 — 12)

a) Use a midpoint Riemann sum with three subintervals to approximate $\int_0^{12} R(t)dt$, and explain the meaning of this definite integral in terms of the water flow, using correct units.



N=3 3 rectangle

$$\int_0^{12} R(t)\,dt = 4\cdot R(2) + 4R(6) + 4R(10)$$
$$= 4(\underline{13.4}) + 4(14.3) + 4(14.8)$$
$$= 170 \text{ gallons}$$

b) A model for the rate of water flow is given by the function: $P(t) = \frac{1}{60}(750 + 24t - t^2)$ where the positive rate P is measured in gallons per hour and the time t is measured in hours. Use *P(t)* to find the average rate of water flow during the 12-hour time period. Indicate units of measure.

$$\text{Ave Value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$* = \frac{1}{12-0}\int_0^{12} \frac{1}{60}(750+24t-t^2)\,dt$$
$$= \frac{1}{12}\cdot\frac{1}{60}\int_0^{12}(750+24t-t^2)\,dt$$
$$= \frac{1}{720}\left[750t + 12t^2 - \frac{t^3}{3}\right]\Bigg|_0^{12}$$
$$= 14.1 \text{ gallons/hr}$$

5. Particle A moves along a horizontal line with a velocity $V_A(t)$, where $V_A(t)$ is a positive continuous function of t. The time t is measured in seconds, and the velocity is measured in cm/sec. The velocity $V_A(t)$, of the particle at selected times is given in the table below.

t(sec)	0	2	5	7	10
$V_A(t)$ (cm/sec)	1.7	6.8	7.4	15.6	24.9

(0, 2) (2, 5) (5, 7) (7, 10)

a) Use data from the table to approximate the distance traveled by particle A over the interval $0 \le t \le 10$ seconds by using a right Riemann sum with four subintervals. Indicate units of measure.

N = 4 Rectangle

$D = V \cdot t$

$D = 6.8(2) + 7.4(3) + 15.6(2) + 24.9(3)$

$D = 141.7$ cm

b) Particle B moves along the same line with an acceleration of $a_B(t) = 2t - 7$ cm/sec². At time t=1 second, the velocity of particle B is 13 cm/sec. Which particle is traveling faster at time t=5 seconds?

$t = 5 \quad V_A = 7.4$ cm/sec

$V_B = \int 2t - 7 \, dt$

$V_B = t^2 - 7t + C$

$t = 1 \quad V_B = 13$

$13 = 1^2 - 7(1) + C$

$19 = C$

$V_B = t^2 - 7t + 19$

$t = 5 \quad V_B = 5^2 - 7(5) + 19$

$V_B = 9$

$\begin{array}{ccc} V_A & < & V_B \\ 7.4 & & 9 \end{array}$